01 Campo conservativo

$\nabla\varphi = \vec{F}$ → função potencial

$\int_\gamma \vec{F}\, d\vec{r} = \varphi(\gamma(b)) - \varphi(\gamma(a))$ → independência do caminho

$\int_\gamma \vec{F}\, d\vec{r} = 0$ → curva fechada

$\text{rot}\vec{F} = \vec{0}$

$$\text{rot}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$$

- $\vec{F}(x,y) = \dfrac{-y}{x^2+y^2}\hat{i} + \dfrac{x}{x^2+y^2}\hat{j}$

$P = \dfrac{-y}{x^2+y^2} \quad Q = \dfrac{x}{x^2+y^2} \quad R = 0$

- Rotacional

$$\text{rot}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y/x^2+y^2 & x/x^2+y^2 & 0 \end{vmatrix}$$

$$= 0\hat{i} + \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)\hat{k} + \frac{\partial}{\partial z}\left(\frac{-y}{x^2+y^2}\right)\hat{j}$$

$$-\frac{\partial}{\partial z}\left(\frac{x}{x^2+y^2}\right)\hat{i} + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\hat{k} - 0\hat{j}$$

$$= \frac{(x^2+y^2) - x.2x}{(x^2+y^2)^2}\hat{k} + \frac{(x^2+y^2) - y.2y}{(x^2+y^2)^2}\hat{k}$$

$$= \frac{x^2+y^2}{(x^2+y^2)^2}(-2x^2 - 2y^2)\hat{k}$$

$$= \frac{2}{x^2+y^2}(-x^2 - y^2)\hat{k}$$

$\text{rot}\vec{F} \neq \vec{0}$

$\therefore \vec{F}(x,y) = \dfrac{-y}{x^2+y^2}\hat{i} + \dfrac{x}{x^2+y^2}\hat{j}$ não é conservativo

02. Calcular

· Analisar se campo é conservativo

$\vec{F}(x,y,z) = (x^2+y, -7yz, 2xz^2)$

$$\text{rot}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2+y & -7yz & 2xz^2 \end{vmatrix}$$

$$= \frac{\partial}{\partial y}(2xz^2)\hat{i} - \frac{\partial}{\partial x} 7yz\,\hat{k} + \frac{\partial}{\partial z}(x^2+y)\hat{j}$$

$$\frac{\partial}{\partial z} 7yz\,\hat{i} - \frac{\partial}{\partial y}(x^2+y)\hat{k} - \frac{\partial}{\partial x} 2xz^2\,\hat{j}$$

$$7y\hat{i} - 1\hat{k} - 2z^2\hat{j}$$

$\text{rot}\vec{F} \neq \vec{0}$ ∴ $\vec{F}$ não é conservativo

$\int \vec{F}\cdot d\vec{r} = \int_a^b \langle \vec{F}, \gamma'(t) \rangle dt = \partial$

a) $\gamma_1 = (t,t,t)$, $0 \le t \le 1$

$\gamma_1' = (1,1,1)$

$\vec{F}(\gamma(t)) = (t^2+t, -7t^2, 2t^3)$

$$\int_\gamma (x^2+y)dx - 7yz\,dy + 2xz^2 dz = \int_0^1 \langle (t^2+t, -7t^2, 2t^3)\cdot(1,1,1)\rangle dt$$

$$= \int_0^1 (t^2+t-7t^2+2t^3)\,dt$$

$$= \int_0^1 (2t^3-6t^2+t)\,dt$$

$$= \left[\frac{2t^4}{4} - \frac{6t^3}{3} + \frac{t^2}{2}\right]_0^1$$

$$= \frac{1}{2} - 2 + \frac{1}{2}$$

$$= \boxed{-1}$$

b) $\gamma_2 = (t, t^2, t^3) \quad 0 \leq t \leq 1$

$\gamma'_2 = (1, 2t, 3t^2)$

$\vec{F}(\gamma(t)) = (t^2 + t^2, -7t^2 t^3, 2t\, t^6)$

$\vec{F}(\gamma(t)) = (2t^2, -7t^5, 2t^7)$

$\mathcal{G} = \int_\gamma \vec{F} \cdot d\vec{r} = \int_a^b \langle \vec{F}, \gamma' \rangle \, dt$

$$\int_\gamma (x^2+y)dx - 7yz\,dy + 2xz^2\,dz = \int_0^1 (2t^2, -7t^5, 2t^7)\cdot(1, 2t, 3t^2)\,dt$$

$$= \int_0^1 (2t^2, -14t^6, 6t^9)\,dt$$

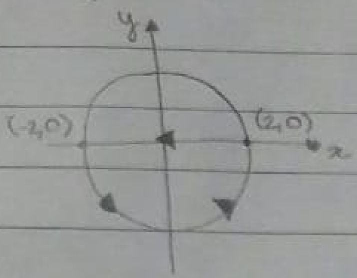
$$= \left[2\frac{t^3}{3} - 14\frac{t^7}{7} + \frac{6t^{10}}{10} \right]_0^1$$

$$= \frac{2}{3} - 2 + \frac{3}{5}$$

$$= \boxed{\frac{-11}{15}}$$

04. $\int_\gamma \cos x\, dx + (2y^{\cos y} + y)\, dy$

$x^2 + y^2 = 4 \rightarrow$ círculo de raio $= 2$



sentido anti-horário

$\vec{F}(x,y) = (P(x,y) ; Q(x,y))$

$\begin{cases} P = \cos x \\ Q = 2y^{\cos y} \end{cases}$

$\frac{d}{dx} a^x = a^x \ln a$

$2y^{\cos y} \rightarrow 2y^{\cos y} \ln y$

• Derivadas Parciais

$\frac{\partial Q}{\partial x} = 0 \qquad \frac{\partial P}{\partial y} = 0$

• Teorema de Green

$\int_{\partial D} P dx + Q dy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy$

$\int_\gamma \cos x\, dx + (2y^{\cos y} + y)\, dy = \int_0^{-2} \int_2^{-2} 0\, dx\, dy$

$\boxed{\int_\gamma \cos x\, dx + (2y^{\cos y} + y)\, dy = 0}$

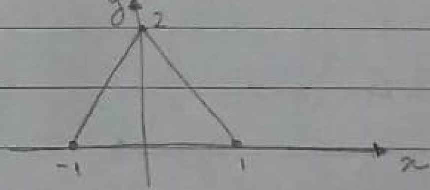
01 Teorema de Green

$\vec{F}(x,y) = 4x^3y^3\, \hat{i} + (3x^4y^2 + 5x)\, \hat{j} \qquad \vec{F}(x,y) = (P(x,y) ; Q(x,y))$

γ - fronteira do triângulo de vértices $(-1,0)$, $(1,0)$ e $(0,2)$

• Derivadas Parciais

$\frac{\partial Q}{\partial x} = 12x^3y^2 + 5 \qquad \frac{\partial P}{\partial y} = 12x^3y^2$



• Teorema de Green

$\oint_\gamma \vec{F} \cdot d\gamma = \iint_\delta \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy$

$\oint_\gamma \vec{F} \cdot d\gamma = \iint (12x^3y^2 + 5 - 12x^3y^2)\, dx\, dy = \iint 5\, dx\, dy = 5\, \text{Área} = 5 \cdot \frac{2 \cdot 2}{2}$

$\boxed{\oint \vec{F}\, d\gamma = 10}$

02 Teorema da Divergência no Plano

fluxo de $\vec{F}$

fronteira do retângulo $4 \le x \le 9$, $5 \le y \le 9$

$\vec{F}(x,y) = \sqrt{x}\,\vec{i} + y^2\vec{j}$ $\qquad \vec{F}(x,y) = (P(x,y) ; Q(x,y))$

* EQUAÇÃO

· Divergente

$$\operatorname{div}\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

$$\operatorname{div}\vec{F} = \frac{1}{2}x^{-1/2} + 2y$$

· Teorema da Divergência no Plano

$$\oint_{\gamma} \vec{F}.\vec{n}\,ds = \iint \operatorname{div}\vec{F}\,dx\,dy$$

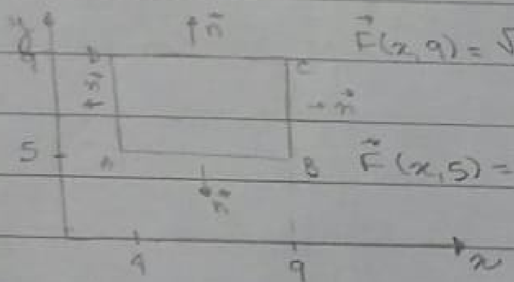
$$\oint_{\gamma} \vec{F}.\vec{n}\,ds = \int_9^5\int_4^9 \left(\frac{1}{2}x^{-1/2} + 2y\right)dx\,dy = \int_9^5 \left[x^{1/2} + 2xy\right]\Big|_4^9 dy$$

$$\oint_{\gamma} \vec{F}.\vec{n}\,ds = \int_9^5 \left[(9^{1/2} + 2.9.y) - (4^{1/2} + 2.4\,y)\right]dy = \int_9^5 (1 + 10y)\,dy$$

$$\oint_{\gamma} \vec{F}.\vec{n}\,ds = \left[y + \frac{10y^2}{2}\right]\Big|_9^5 = \left(9 + \frac{10.9^2}{2}\right) - \left(5 + \frac{10.5^2}{2}\right)$$

$$\boxed{\oint_{\gamma} \vec{F}.\vec{n}\,ds = 284}$$

* GEOMETRIA



$\vec{F}(x,9) = \sqrt{x}\,\vec{i} + 81\vec{j}$ $\qquad \vec{F}(4,y) = 2\vec{i} + y^2\vec{j}$

$\vec{F}(x,5) = \sqrt{x}\,\vec{i} + 25\vec{j}$ $\qquad \vec{F}(9,y) = 3\vec{i} + y^2\vec{j}$

$AB = (\sqrt{x}\,\vec{i} + 25\vec{j}).(0,\vec{j})\cos 180° = -25$

$CD = (\sqrt{x}\,\vec{i} + 81\vec{j}).(0,\vec{j}) = 81$

$AD = (2\vec{i} + y^2\vec{j}).(\vec{i},0)\cos 180° = -2$

$BC = (3\vec{i} + y^2\vec{j}).(\vec{i},0) = 3$

FLUXO $= -25.5 + 81.5 - 2.4 + 3.4$

FLUXO $= -125 + 405 - 8 + 12$

$\boxed{\text{FLUXO} = 284}$

03. Integração em campos conservativos

$$\int_\gamma x\,dx + y\,dy$$

$$\gamma(t) = \begin{cases} x = \operatorname{arctg} t \\ y = \operatorname{sen} t^3 \end{cases} \quad 0 \le t \le 1$$

· Função Potencial

$$\vec{\nabla}\varphi = \vec{F} = \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}\right)$$

$$\left.\begin{array}{l} \dfrac{\partial \varphi}{\partial x} = x \rightarrow \dfrac{x^2}{2} \\ \dfrac{\partial \varphi}{\partial y} = y \rightarrow \dfrac{y^2}{2} \end{array}\right\} \varphi = \frac{x^2+y^2}{2} = \frac{\operatorname{arctg}^2 t + (\operatorname{sen} t^3)^2}{2}$$

· $\int_\gamma \vec{F}\cdot d\gamma = \int_\gamma \nabla\varphi \cdot d\gamma = \varphi(B) - \varphi(A)$

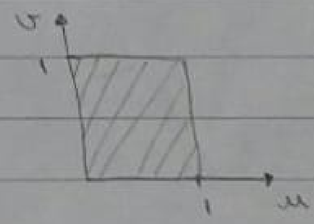
$$\int_\gamma x\,dx + y\,dy = \frac{\operatorname{arctg}^2 1 + \operatorname{sen}^2 1}{2} - \frac{\operatorname{arctg}^2 0 + \operatorname{sen} 0}{2}$$

$$\int_\gamma x\,dx + y\,dy = \frac{(\pi/4)^2 + \operatorname{sen}^2 1}{2}$$

$$\boxed{\int_\gamma x\,dx + y\,dy = \frac{\pi^2 + 16\operatorname{sen}^2 1}{32}}$$

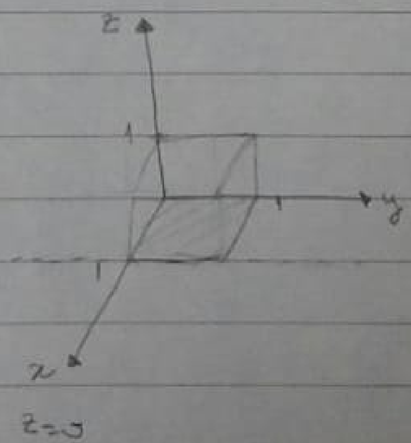
04. Área de Superfície

$\sigma(u,v) = (1, u, v) \quad 0 \le u \le 1 \quad 0 \le v \le 1$



· Derivadas Parciais

$$\frac{\partial \sigma}{\partial u} = (0,1,0) \qquad \frac{\partial \sigma}{\partial v} = (0,0,1)$$



· Produto vetorial

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \vec{i}$$

· Norma

$$\left\| \frac{\partial \sigma}{\partial u} \times \frac{\partial \sigma}{\partial v} \right\| = 1$$

· Área

$$\text{Área } \sigma = \iint_K \left\| \frac{\partial \sigma}{\partial u} \times \frac{\partial \sigma}{\partial v} \right\| du\,dv$$

$$\text{Área } \sigma = \int_0^1 \int_0^1 1 \, du\,dv \qquad \boxed{\text{Área } \sigma = 1}$$

03. $\displaystyle\int_\gamma \frac{x}{x^2+y^2}\,dx + \frac{y}{x^2+y^2}\,dy$ $\qquad \gamma(A) = (2,0)$, $\gamma(B) = (1,0)$

· Derivadas Parciais

$$\frac{\partial Q}{\partial x} = \frac{-y \cdot 2x}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial P}{\partial y} = \frac{-x \, 2y}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

· Rotacional

$$\text{rot } \vec{F} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{-2xy}{(x^2+y^2)^2} - \left(\frac{-2xy}{(x^2+y^2)^2}\right) = 0 \rightarrow \text{CONDIÇÃO NECESSÁRIA}$$

· Função Potencial

$$\nabla \varphi = \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}\right) = \vec{F}$$

$$\begin{cases} \dfrac{\partial \varphi}{\partial x} = \dfrac{x}{x^2+y^2} & (1) \\ \dfrac{\partial \varphi}{\partial y} = \dfrac{y}{x^2+y^2} & (2) \end{cases}$$

De (1) $\displaystyle \varphi = \int \frac{x}{x^2+y^2}\,dx \qquad x^2+y^2 = u, \quad du = 2x\,dx$

$$\varphi = \int \frac{x}{u}\frac{du}{2x} = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln u + C = \frac{1}{2}\ln(x^2+y^2) + C$$

$$\frac{\partial \varphi}{\partial y} = \frac{1}{2}\frac{2y}{(x^2+y^2)} = \frac{y}{x^2+y^2} = (2) \qquad \therefore C = 0$$

$$\therefore \varphi = \frac{1}{2}\ln(x^2+y^2)$$

· $\displaystyle\int_\gamma \vec{F} \cdot d\gamma = \int_\gamma \nabla\varphi \, d\gamma = \varphi(B) - \varphi(A)$

$$\int_\gamma \frac{x}{x^2+y^2}\,dx + \frac{y}{x^2+y^2}\,dy = \frac{1}{2}\ln(1^2+0^2) - \frac{1}{2}\ln(2^2+0^2)$$

$$\int_\gamma \frac{x}{x^2+y^2}\,dx + \frac{y}{x^2+y^2}\,dy = \cancel{\frac{1}{2}\ln 1} - \frac{1}{2}\ln 4$$

$$\boxed{\int_\gamma \frac{x}{x^2+y^2}\,dx + \frac{y}{x^2+y^2}\,dy = -\frac{1}{2}\ln 4} \rightarrow -\ln 2$$