1 First Order Equations

Q1. $\dfrac{dx}{dt} = 4(x^2+1);\ x\left(\dfrac{\pi}{4}\right) = 1.$

Solution. This is a separable equation, so we proceed using the method of separation of variables.

$$\begin{aligned} \frac{dx}{dt} &= 4(x^2+1) \\ \frac{dx}{x^2+1} &= 4dt \\ \int \frac{dx}{x^2+1} &= \int 4dt \\ \arctan x &= 4t + C. \end{aligned}$$

Here we utilize our initial condition to determine the value of C.

$$\begin{aligned} C &= \arctan x - 4t \\ &= \arctan(1) - 4\left(\frac{\pi}{4}\right) \\ &= \frac{\pi}{4} - \pi \\ &= \frac{-3\pi}{4}. \end{aligned}$$

We now finish our derivation of an explicit solution.

$$\begin{aligned} \arctan x &= 4t - \frac{3\pi}{4} \\ &\boxed{x(t) = \tan\left(4t - \frac{3\pi}{4}\right)} \end{aligned}$$

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Q2. $\dfrac{dy}{dx} - (\sin x)y = 2\sin x$; $y\left(\dfrac{\pi}{2}\right) = 1$.

Solution. This is a linear equation, so we proceed by finding an integrating factor. Note this equation is already in standard form. Let μ denote the integrating factor. Then

$$\begin{aligned} \mu &= \mathrm{e}^{\int P(x)dx} \\ &= \mathrm{e}^{\int -\sin x dx} \\ &= \mathrm{e}^{\cos x}. \end{aligned}$$

Multiplying our differential equation by the integrating factor, we can solve for a general solution.

$$\begin{aligned} \mathrm{e}^{\cos x}\frac{dy}{dx} - \sin x\,\mathrm{e}^{\cos x}\,y &= 2\sin x\,\mathrm{e}^{\cos x} \\ (\mathrm{e}^{\cos x}\,y)' &= 2\sin x\,\mathrm{e}^{\cos x} \\ \mathrm{e}^{\cos x}\,y &= \int 2\sin x\,\mathrm{e}^{\cos x}\,dx \\ \mathrm{e}^{\cos x}\,y &= -2\,\mathrm{e}^{\cos x} + C \\ y &= -2 + C\,\mathrm{e}^{-\cos x}. \end{aligned}$$

Finally, we make use of our initial condition to determine the value of C and derive a solution to the IVP.

$$\begin{aligned} 1 &= -2 + C\,\mathrm{e}^{-\cos(\pi/2)} \\ 3 &= C\,\mathrm{e}^{0} \\ 3 &= C. \end{aligned}$$

$$\boxed{y(x) = -2 + 3\,\mathrm{e}^{-\cos x}}$$

$\diamondsuit$

Q3. $(x+y)^2dx + (2xy + x^2 - 1)dy = 0$; $y(1) = 1$.

Solution. This equation appears to be exact, so we check to be sure.

$$\begin{aligned}
(x+y)^2dx + (2xy + x^2 - 1)dy &= 0\\
(x^2 + 2xy + y^2)dx + (2xy + x^2 - 1)dy &= 0\\
M_y = \frac{\partial}{\partial y}(x^2 + 2xy + y^2) &= 2x + 2y\\
N_x = \frac{\partial}{\partial x}(2xy + x^2 - 1) &= 2y + 2x\\
M_y &= N_x.
\end{aligned}$$

Therefore, this is an exact equation so we proceed using that method.

$$\begin{aligned}
F(x,y) &= \int M dx\\
&= \int (x^2 + 2xy + y^2)dx\\
&= \frac{1}{3}x^3 + x^2y + xy^2 + h(y).
\end{aligned}$$

Using N, we solve for $h(y)$.

$$\begin{aligned}
N &= 2xy + x^2 - 1\\
N &= \frac{\partial}{\partial y}F(x,y) = x^2 + 2xy + h'(y)\\
2xy + x^2 - 1 &= 2xy + x^2 + h'(y)\\
h'(y) &= -1\\
h(y) &= -y + C.
\end{aligned}$$

Putting these together, we have a general solution to the equation.

$$\frac{1}{3}x^3 + x^2y + xy^2 - y = C.$$

Using our initial condition, we solve for the value of C and derive a solution to the IVP.

$$C = \frac{1}{3}(1)^3 + (1)^2(1) + (1)(1)^2 - 1 = \frac{4}{3}$$

$$\boxed{\frac{1}{3}x^3 + x^2y + xy^2 - y = \frac{4}{3}}$$

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Q4. $(2y^2 + 3x)dx + 2xydy = 0$.

Solution. This equation appears to be exact, so we check to be sure.

$$\begin{aligned} M_y &= \frac{\partial}{\partial y}(2y^2 + 3x) = 4y \\ N_x &= \frac{\partial}{\partial x}(2xy) = 2y \\ M_y &\neq N_x. \end{aligned}$$

Therefore, this is not an exact equation. We check for a possible integrating factor to make it exact.

$$\begin{aligned} \frac{M_y - N_x}{N} &= \frac{2y}{2xy} = \frac{1}{x}. \\ \mu &= e^{\int (1/x)dx} = e^{\ln x} = x. \end{aligned}$$

Multiplying the equation by μ should make it exact.

$$\begin{aligned} (2xy^2 + 3x^2)dx + 2x^2ydy = 0 & \\ M_y &= \frac{\partial}{\partial y}(2xy^2 + 3x^2) = 4xy \\ N_x &= \frac{\partial}{\partial x}(2x^2y) = 4xy \\ M_y &= N_x. \end{aligned}$$

The modified equation is exact, so we proceed using that method.

$$\begin{aligned} F(x, y) &= \int N dy \\ &= \int 2x^2ydy \\ &= x^2y^2 + h(x). \end{aligned}$$

Using M, we solve for $h(x)$.

$$\begin{aligned} M &= 2xy^2 + 3x^2 \\ M &= \frac{\partial}{\partial x}F(x, y) = 2xy^2 + h'(x) \\ 2xy^2 + 3x^2 &= 2xy^2 + h'(x) \\ h'(x) &= 3x^2 \\ h(x) &= x^3 + C. \end{aligned}$$

Putting these together,we have a general solution to the equation.

$$\boxed{x^2y^2 + x^3 = C}$$

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Q5. $xy^2\dfrac{dy}{dx} = y^3 - x^3$; $y(1) = 2$.

Solution. We verify that this equation is homogeneous.

$$\begin{aligned}
xy^2\frac{dy}{dx} &= y^3 - x^3 \\
xy^2 dy &= (y^3 - x^3)dx \\
(x^3 - y^3)dx + xy^2 dy &= 0 \\
M(tx, ty) &= (tx)^3 - (ty)^3 = t^3(x^3 - y^3) = t^3 M(x, y) \\
N(tx, ty) &= (tx)(ty)^2 = t^3 xy^2 = t^3 N(x, y).
\end{aligned}$$

Therefore, this is a homogeneous equation, and so we make the substitution $y = ux$.

$$\begin{aligned}
x(ux)^2\left(u + x\frac{du}{dx}\right) &= (ux)^3 - x^3 \\
u^3x^3 + u^2x^4\frac{du}{dx} &= u^3x^3 - x^3 \\
u^2x^4\frac{du}{dx} &= -x^3.
\end{aligned}$$

The resulting equation is separable, so we proceed using the method of separation of variables.

$$\begin{aligned}
u^2x^4\frac{du}{dx} &= -x^3 \\
u^2 du &= -\frac{1}{x}dx \\
\int u^2 du &= -\int \frac{1}{x}dx \\
\frac{1}{3}u^3 &= -\ln x + C \\
\frac{1}{3}\left(\frac{y}{x}\right)^3 &= -\ln x + C \\
y^3 &= -3x^3\ln x + Cx^3.
\end{aligned}$$

Finally, using our initial condition, we determine the value of C to obtain a solution to the IVP.

$$\begin{aligned}
2^3 &= -3(1)^3\ln(1) + C(1)^3 \\
8 &= C
\end{aligned}$$

$$\boxed{y^3 = -3x^3\ln x + 8x^3}$$

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Q6. $x^2\dfrac{dy}{dx} - 2xy = 3y^4$; $y(1) = \dfrac{1}{2}$.

Solution. This is a Bernoulli equation, so we make the substitution $u = y^{-3}$.

$$\begin{aligned} x^2\frac{dy}{dx} - 2xy &= 3y^4 \\ -3x^2y^{-4}\frac{dy}{dx} + 6xy^{-3} &= -9 \\ x^2\frac{du}{dx} + 6xu &= -9 \\ \frac{du}{dx} + \frac{6}{x}u &= \frac{-9}{x^2}. \end{aligned}$$

The resulting equation is linear, so we find an integrating factor. Note we have already put the equation into standard form.

$$\mu = \mathrm{e}^{\int P(x)dx} = \mathrm{e}^{\int (6/x)dx} = \mathrm{e}^{6\ln x} = x^6.$$

Multiplying by the integrating factor, we can solve for a general solution to the equation.

$$\begin{aligned} x^6\frac{du}{dx} + 6x^5u &= -9x^4 \\ (x^6u)' &= -9x^4 \\ x^6u &= \frac{-9}{5}x^5 + C \\ u &= \frac{-9}{5}x^{-1} + Cx^{-6} \\ y^{-3} &= \frac{-9}{5}x^{-1} + Cx^{-6}. \end{aligned}$$

Utilizing our initial condition, we can determine the value of C and derive a solution to the IVP.

$$\begin{aligned} \left(\frac{1}{2}\right)^{-3} &= \frac{-9}{5}(1)^{-1} + C(1)^{-6} \\ 8 &= \frac{-9}{5} + C \\ C &= \frac{49}{5} \\ &\boxed{y^{-3} = \frac{-9}{5}x^{-1} + \frac{49}{5}x^{-6}} \end{aligned}$$

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Q7. $\dfrac{dy}{dx} = \cos(x+y)$; $y(0) = \dfrac{\pi}{4}$.

Solution. This equation is of the form $\dfrac{dy}{dx} = f(x+y)$, and so we make the substitution $u = x + y$.

$$\begin{aligned} \frac{dy}{dx} &= \cos(x+y) \\ -1 + \frac{du}{dx} &= \cos u \\ \frac{du}{dx} &= \cos u + 1. \end{aligned}$$

The resulting equation is separable, so we proceed using the method of separation of variables.

$$\begin{aligned} \frac{du}{dx} &= \cos u + 1 \\ \frac{du}{\cos u + 1} &= dx \\ \int \frac{du}{\cos u + 1} &= \int dx \\ \int \frac{\cos u - 1}{\cos^2 u - 1} du &= x + C \\ \int \frac{1}{\sin^2 u} du - \int \frac{\cos u}{\sin^2 u} du &= x + C \\ \int \csc^2 u du - \int \csc u \cot u du &= x + C \\ \csc u - \cot u &= x + C \\ \csc(x+y) - \cot(x+y) &= x + C. \end{aligned}$$

Using our initial condition, we can solve for the value of C and derive a solution to the IVP.

$$\begin{aligned} \csc\left(0 + \frac{\pi}{4}\right) - \cot\left(0 + \frac{\pi}{4}\right) &= 0 + C \\ C &= \frac{1}{\sin(\pi/4)} - \frac{\cos(\pi/4)}{\sin(\pi/4)} \\ &= \sqrt{2} - 1 \end{aligned}$$

$$\boxed{\csc(x+y) - \cot(x+y) = x + \sqrt{2} - 1}$$

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2 Higher Order Equations

Q8. Using reduction of order, determine a second solution to the given differential equation.

$$xy'' + y' = 0; \;\; y_1 = \ln x.$$

Solution. In reduction of order, we seek a solution y_2 which is of the form $y_2 = u(x)y_1(x)$.

$$\begin{aligned} y_2 &= u \ln x \\ y_2' &= u' \ln x + \frac{u}{x} \\ y_2'' &= u'' \ln x + 2\frac{u'}{x} - \frac{u}{x^2}. \end{aligned}$$

By construction, y_2 will be a solution to the given differential equation.

$$\begin{aligned} xy_2'' + y_2' &= 0 \\ xu'' \ln x + 2u' - \frac{u}{x} + u' \ln x + \frac{u}{x} &= 0 \\ x \ln x u'' + (\ln x + 2)u' &= 0. \end{aligned}$$

We now make the substitution $w = u'$.

$$\begin{aligned} x \ln x u'' + (\ln x + 2)u' &= 0 \\ x \ln x w' + (\ln x + 2)w &= 0. \end{aligned}$$

The resulting equation is separable, so we proceed using the method of separation of variables.

$$\begin{aligned} x \ln x \frac{dw}{dx} + (\ln x + 2)w &= 0 \\ x \ln x \frac{dw}{dx} &= -(\ln x + 2)w \\ \frac{dw}{w} &= \frac{-\ln x - 2}{x \ln x} dx \\ \int \frac{dw}{w} &= -\int \frac{\ln x + 2}{x \ln x} dx \\ \ln w &= -\int \frac{dx}{x} - 2\int \frac{dx}{x \ln x} \\ \ln w &= -\ln x - 2\ln(\ln x) + C_1 \\ \ln w &= -\ln x(\ln x)^2 + C_1 \\ w &= \frac{C_1}{x(\ln x)^2} \\ u' &= \frac{C_1}{x(\ln x)^2} \\ u &= -\frac{C_1}{\ln x} + C_2. \end{aligned}$$

Let $C_1 = -1$ and $C_2 = 0$.

$$\begin{aligned} u &= \frac{1}{\ln x} \\ &\boxed{y_2 = 1} \end{aligned}$$

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Q9. $y^{(4)} - y = 0.$

Solution. This is a linear fourth-order equation with constant coefficients, so we look for solutions of the form $y = \mathrm{e}^{mx}$.

$$\begin{aligned} m^4 \mathrm{e}^{mx} - \mathrm{e}^{mx} &= 0 \\ m^4 - 1 &= 0 \\ (m^2-1)(m^2+1) &= 0 \\ (m-1)(m+1)(m^2+1) &= 0 \\ m &= \pm 1, \pm i \\ &\boxed{y(x) = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x} + c_3 \cos x + c_4 \sin x} \end{aligned}$$

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Q10. $y'' + 4y = 3 \sin 2x.$

Solution. This is a linear second-order nonhomogeneous equation with constant coefficients. We first solve the homogeneous equation.

$$\begin{aligned} y_c'' + 4y_c &= 0 \\ m^2 \mathrm{e}^{mx} + 4 \mathrm{e}^{mx} &= 0 \\ m^2 + 4 &= 0 \\ m &= \pm 2i \\ y_c &= c_1 \cos 2x + c_2 \sin 2x. \end{aligned}$$

We use the method of undetermined coefficients to determine a particular solution y_p. Since $\sin x$ is a solution to the homogeneous equation, we modify the usual trial solution for $\sin x$ by a factor of x.

$$\begin{aligned} y_p &= Ax \cos 2x + Bx \sin 2x \\ y_p' &= A \cos 2x - 2Ax \sin 2x + B \sin 2x + 2Bx \cos 2x \\ y_p'' &= -4A \sin 2x - 4Ax \cos 2x + 4B \cos 2x - 4Bx \sin 2x. \end{aligned}$$

By construction, y_p is a solution to the given differential equation.

$$\begin{aligned} y_p'' + 4y_p &= 3 \sin 2x \\ -4A \sin 2x + 4B \cos 2x &= 3 \sin 2x \\ -4A = 3 \Rightarrow A &= -\frac{3}{4} \\ 4B = 0 \Rightarrow B &= 0 \\ y_p &= -\frac{3}{4} x \cos 2x. \end{aligned}$$

Putting these together, we obtain a general solution to the given differential equation.

$$\boxed{y(x) = c_1 \cos 2x + c_2 \sin 2x - \frac{3}{4} x \cos 2x}$$

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Q11. $y''' + y' = \tan x$.

Solution. This is a linear third-order nonhomogeneous equation with constant coefficients. We first solve the homogeneous equation.

$$\begin{aligned}
y_c''' + y_c' &= 0\\
m^3 \mathrm{e}^{mx} + m\,\mathrm{e}^{mx} &= 0\\
m^3 + m &= 0\\
m(m^2+1) &= 0\\
m &= 0, \pm i\\
y_c &= c_1 + c_2 \cos x + c_3 \sin x.
\end{aligned}$$

We use the method of variation of parameters to determine a particular solution y_p. We first require the Wronskian. Set $y_1 = 1$, $y_2 = \cos x$, and $y_3 = \sin x$.

$$W = \det\begin{pmatrix} y_1 & y_2 & y_3\\ y_1' & y_2' & y_3'\\ y_1'' & y_2'' & y_3''\end{pmatrix} = \det\begin{pmatrix} 1 & \cos x & \sin x\\ 0 & -\sin x & \cos x\\ 0 & -\cos x & -\sin x\end{pmatrix} = 1.$$

We now use Cramer's Rule to find functions u_1, u_2, and u_3 so that $y_p = u_1y_1 + u_2y_2 + u_3y_3$ is a solution to the given differential equation.

$$W_1 = \det\begin{pmatrix} 0 & \cos x & \sin x\\ 0 & -\sin x & \cos x\\ \tan x & -\cos x & -\sin x\end{pmatrix} = \tan x \qquad W_2 = \det\begin{pmatrix} 1 & 0 & \sin x\\ 0 & 0 & \cos x\\ 0 & \tan x & -\sin x\end{pmatrix} = -\sin x \qquad W_3 = \det\begin{pmatrix} 1 & \cos x & 0\\ 0 & -\sin x & 0\\ 0 & -\cos x & \tan x\end{pmatrix} = -\sin x \tan x$$

We now have $u_1' = \frac{W_1}{W}$, $u_2' = \frac{W_2}{W}$, and $u_3' = \frac{W_3}{W}$.

$$\begin{aligned}
u_1 &= \int \tan x dx\\
&= \int \frac{\sin x}{\cos x} dx\\
&= -\ln|\cos x|
\end{aligned}
\qquad
\begin{aligned}
u_2 &= \int -\sin x dx\\
&= \cos x
\end{aligned}
\qquad
\begin{aligned}
u_3 &= \int -\sin x \tan x dx\\
&= \int \frac{-\sin^2 x}{\cos x} dx\\
&= \int \frac{\cos^2 x - 1}{\cos x} dx\\
&= \int \cos x dx - \int \sec x dx\\
&= \sin x - \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx\\
&= \sin x - \ln|\sec x + \tan x|
\end{aligned}$$

We can put all of this together to determine y_p and a general solution to the differential equation.

$$\begin{aligned}
y_p &= u_1y_1 + u_2y_2 + u_3y_3\\
&= -\ln|\cos x| + \cos^2 x + \sin^2 x - \sin x \ln|\sec x + \tan x|\\
&= 1 - \ln|\cos x| - \sin x \ln|\sec x + \tan x|
\end{aligned}$$

$$\boxed{y(x) = c_1 + c_2 \cos x + c_3 \sin x - \ln|\cos x| - \sin x \ln|\sec x + \tan x|}$$

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