Q25. Determine the first 7 terms of a solution to the following differential equation.

$$(x^3+1)y''' + (x^2-2)y'' + xy' + y = 0;\ y(0)=2,\ y'(0)=1,\ y''(0)=0.$$

Solution. We search for a solution given by a power series centered at $x=0$.

$$y = \sum_{n=0}^{\infty} c_n x^n;\ y' = \sum_{n=1}^{\infty} nc_n x^{n-1};\ y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2};\ y''' = \sum_{n=3}^{\infty} n(n-1)(n-2)c_n x^{n-3}$$

$$(x^3+1)\sum_{n=3}^{\infty} n(n-1)(n-2)c_n x^{n-3} + (x^2-2)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=1}^{\infty} nc_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=3}^{\infty} n(n-1)(n-2)c_n x^n + \sum_{n=3}^{\infty} n(n-1)(n-2)c_n x^{n-3} + \sum_{n=2}^{\infty} n(n-1)c_n x^n - 2\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} nc_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0.$$

We now make appropriate substitutions for the indices on each sum so as to be left with x^k in the summand.

$$\sum_{k=3}^{\infty} k(k-1)(k-2)c_k x^k + \sum_{k=0}^{\infty}(k+3)(k+2)(k+1)c_{k+3}x^k + \sum_{k=2}^{\infty} k(k-1)c_k x^k$$
$$-2\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=1}^{\infty} kc_k x^k + \sum_{k=0}^{\infty} c_k x^k = 0.$$

Next we combine the six sums into a single sum by rewriting each sum so they start at $k=3$.

$$\sum_{k=3}^{\infty} k(k-1)(k-2)c_k x^k + \left(6c_3 + 24c_4 x + 60c_5 x^2 + \sum_{k=3}^{\infty}(k+3)(k+2)(k+1)c_{k+3}x^k\right) + \left(2c_2 x^2 + \sum_{k=3}^{\infty} k(k-1)c_k x^k\right)$$
$$-2\left(2c_2 + 6c_3 x + 12c_4 x^2 + \sum_{k=3}^{\infty}(k+2)(k+1)c_{k+2}x^k\right) + \left(c_1 x + 2c_2 x^2 + \sum_{k=3}^{\infty} kc_k x^k\right)$$
$$+\left(c_0 + c_1 x + c_2 x^2 + \sum_{k=3}^{\infty} c_k x^k\right) = 0$$

$$0 = (6c_3 - 4c_2 + c_0) + (24c_4 - 12c_3 + 2c_1)x + (60c_5 + 5c_2 - 24c_4)x^2$$
$$+\sum_{k=3}^{\infty}[k(k-1)(k-2)c_k + (k+3)(k+2)(k+1)c_{k+3} + k(k-1)c_k - 2(k+2)(k+1)c_{k+2} + kc_k + c_k]x^k.$$

We now make use of our initial conditions, which gives us $c_0 = y(0) = 2$, $c_1 = y'(0) = 1$, and $c_2 = \frac{1}{2}y''(0) = 0$. When two polynomials are equal, their coefficients are equal. So we can use the constant, linear, and quadratic terms of our derived equation to determine the values of c_3, c_4, and c_5.

$$6c_3 - 4c_2 + c_0 = 0 \Rightarrow 6c_3 = -2 \Rightarrow c_3 = -\frac{1}{3}$$
$$24c_4 - 12c_3 + 2c_1 = 0 \Rightarrow 24c_4 = -6 \Rightarrow c_4 = -\frac{1}{4}$$
$$60c_5 + 5c_2 - 24c_4 = 0 \Rightarrow 60c_5 = -6 \Rightarrow c_5 = -\frac{1}{10}.$$

Finally, we use the power series portion of our derived equation to determine a recurrence relation which will allow us to solve for the value of c_6.

$$k(k-1)(k-2)c_k + (k+3)(k+2)(k+1)c_{k+3} + k(k-1)c_k - 2(k+2)(k+1)c_{k+2} + kc_k + c_k = 0$$
$$c_{k+3} = \frac{-c_k[k(k-1)(k-2) + k(k-1) + k + 1] + 2(k+2)(k+1)c_{k+2}}{(k+3)(k+2)(k+1)}$$
$$c_6 = \frac{-16c_3 + 40c_5}{120} = \frac{-2c_3 + 5c_5}{15} = \frac{\frac{2}{3} - \frac{1}{2}}{15} = \frac{1}{90}.$$

Putting this all together, we get a solution to the given differential equation.

$$\boxed{y(x) = 2 + x - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{10}x^5 + \frac{1}{90}x^6 + \cdots}$$

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