Q12. $x^2y'' + xy' + 4y = 0.$

Solution. This is a second-order Cauchy-Euler equation, so we look for solutions of the form $y = x^m$.

$$\begin{aligned} m(m-1)x^m + mx^m + 4x^m &= 0 \\ m^2 + 4 &= 0 \\ m &= \pm 2i \\ \boxed{y(x) = c_1 \cos(2\ln x) + c_2 \sin(2\ln x)} \end{aligned}$$

◊

Q13. $x^2y'' + 5xy' + 4y = 0.$

Solution. This is a second-order Cauchy-Euler equation, so we look for solutions of the form $y = x^m$.

$$\begin{aligned} m(m-1)x^m + 5mx^m + 4x^m &= 0 \\ m^2 + 4m + 4 &= 0 \\ (m+2)^2 &= 0 \\ m &= -2, -2 \\ \boxed{y(x) = c_1x^{-2} + c_2x^{-2}\ln x} \end{aligned}$$

◊

Q14. $xy^{(4)} + 6y''' = 0.$

Solution. We multiply the equation by x^3 to make it into a fourth-order Cauchy-Euler equation.

$$\begin{aligned} xy^{(4)} + 6y''' &= 0 \\ x^4y^{(4)} + 6x^3y''' &= 0 \\ m(m-1)(m-2)(m-3)x^m + 6m(m-1)(m-2)x^m &= 0 \\ m(m-1)(m-2)(m-3) + 6m(m-1)(m-2) &= 0 \\ m(m-1)(m-2)(m+3) &= 0 \\ m &= 0, 1, 2, -3 \\ \boxed{y(x) = c_1 + c_2x + c_3x^2 + c_4x^{-3}} \end{aligned}$$

◊

3 Systems of Equations

Q15. $X' = \begin{pmatrix} 10 & -5 \\ 8 & -12 \end{pmatrix} X.$

Solution. We first determine the eigenvalues of the matrix.

$$\begin{aligned} 0 &= \det \begin{pmatrix} 10-\lambda & -5 \\ 8 & -12-\lambda \end{pmatrix} \\ &= (10-\lambda)(-12-\lambda) + 40 \\ &= \lambda^2 + 2\lambda - 80 \\ &= (\lambda - 8)(\lambda + 10) \\ \lambda &= 8, -10. \end{aligned}$$

We now find an eigenvector for $\lambda = 8$.

$$\begin{aligned} \begin{pmatrix} 2 & -5 \\ 8 & -20 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ 2k_1 &= 5k_2 \\ K_8 &= \begin{pmatrix} 5 \\ 2 \end{pmatrix}. \end{aligned}$$

Next we find an eigenvector for $\lambda = -10$.

$$\begin{aligned} \begin{pmatrix} 20 & -5 \\ 8 & -2 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ 4k_1 &= k_2 \\ K_{-10} &= \begin{pmatrix} 1 \\ 4 \end{pmatrix}. \end{aligned}$$

Putting these together, we get a general solution to the given system.

$$\boxed{X(t) = c_1 \begin{pmatrix} 5 \\ 2 \end{pmatrix} \mathrm{e}^{8t} + c_2 \begin{pmatrix} 1 \\ 4 \end{pmatrix} \mathrm{e}^{-10t}}$$

◇

Q16. $X' = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & -1 \\ 0 & 1 & 0 \end{pmatrix} X.$

Solution. We first determine the eigenvalues of the matrix.

$$\begin{aligned} 0 &= \det \begin{pmatrix} 1-\lambda & 0 & 0 \\ 2 & 2-\lambda & -1 \\ 0 & 1 & -\lambda \end{pmatrix} \\ &= (1-\lambda)[-\lambda(2-\lambda)+1] \\ &= (1-\lambda)(\lambda^2 - 2\lambda + 1) \\ &= (1-\lambda)^3 \\ \lambda &= 1, 1, 1. \end{aligned}$$

We now find an eigenvector for $\lambda = 1$.

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$k_2 = k_3$$

$$2k_1 + k_2 = k_3 \Rightarrow 2k_1 = 0 \Rightarrow k_1 = 0$$

$$K_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$p_2 = p_3 + 1$$

$$2p_1 + p_2 = p_3 + 1 \Rightarrow 2p_1 = 0 \Rightarrow p_1 = 0$$

$$P_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$q_2 = q_3$$

$$2q_1 + q_2 = q_3 + 1 \Rightarrow 2q_1 = 1 \Rightarrow q_1 = \frac{1}{2}$$

$$Q_1 = \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}.$$

Putting these together, we obtain a general solution to the given system.

$$\boxed{X(t) = c_1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^t + c_2 \left[\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right] e^t + c_3 \left[\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \frac{t^2}{2} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \right] e^t}$$

$\diamond$

Q17. $X' = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} X.$

Solution. We first determine the eigenvalues of the matrix.

$$\begin{aligned} 0 &= \det \begin{pmatrix} 3-\lambda & -1 & -1 \\ 1 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{pmatrix} \\ &= (3-\lambda)[(1-\lambda)^2 - 1] + 2(2-\lambda) \\ &= -\lambda(3-\lambda)(2-\lambda) + 2(2-\lambda) \\ &= (\lambda^2 - 3\lambda + 2)(2-\lambda) \\ &= (1-\lambda)(2-\lambda)^2 \\ \lambda &= 1, 2, 2. \end{aligned}$$

We now find an eigenvector for $\lambda = 1$.

$$\begin{aligned} \begin{pmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ k_1 = k_2 &= k_3 \\ K_1 &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \end{aligned}$$

Next we find an eigenvector for $\lambda = 2$.

$$\begin{aligned} \begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ k_1 &= k_2 + k_3 \\ K_2 &= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \\ P_2 &= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}. \end{aligned}$$

Note in this case, we do not need to find generalized eigenvectors for $\lambda = 2$. Putting everything together, we get a general solution to the given system.

$$\boxed{X(t) = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \mathrm{e}^{t} + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \mathrm{e}^{2t} + c_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \mathrm{e}^{2t}}$$

◇

Q18. $X' = \begin{pmatrix} 6 & -1 \\ 5 & 2 \end{pmatrix} X.$

Solution. We first determine the eigenvalues of the matrix.

$$\begin{aligned} 0 &= \det \begin{pmatrix} 6-\lambda & -1 \\ 5 & 2-\lambda \end{pmatrix} \\ &= (6-\lambda)(2-\lambda)+5 \\ &= \lambda^2 - 8\lambda + 17 \\ \lambda &= 4 \pm i. \end{aligned}$$

We now find an eigenvector for $\lambda = 4+i$.

$$\begin{pmatrix} 2-i & -1 \\ 5 & -2-i \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(2-i)k_1 = k_2$$

$$K_{4+i} = \begin{pmatrix} 1 \\ 2-i \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Putting these together, we obtain a general solution to the given system.

$$\boxed{X(t) = c_1 \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cos t - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sin t \right] e^{4t} + c_2 \left[\begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos t + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \sin t \right] e^{4t}}$$

◇

Q19. $X' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} X + \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \mathrm{e}^t.$

Solution. We first solve the homogeneous system by finding the eigenvalues of the matrix.

$$\begin{aligned} 0 &= \det \begin{pmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{pmatrix} \\ &= (1-\lambda)^2 + 1 \\ &= \lambda^2 - 2\lambda + 2 \\ \lambda &= 1 \pm i. \end{aligned}$$

We now find an eigenvector for $\lambda = 1 + i$.

$$\begin{aligned} \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ k_1 &= ik_2 \\ K_{1+i} &= \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix}. \end{aligned}$$

Putting this together, we obtain the complementary solution to this system.

$$X_c = c_1 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \mathrm{e}^t + c_2 \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix} \mathrm{e}^t.$$

We now use the method of variation of parameters to determine a particular solution to this system.

$$\begin{aligned} \Phi &= \begin{pmatrix} \mathrm{e}^t \cos t & \mathrm{e}^t \sin t \\ \mathrm{e}^t \sin t & -\mathrm{e}^t \cos t \end{pmatrix} \\ \det \Phi &= -e^{2t} \\ \Phi^{-1} &= \begin{pmatrix} \mathrm{e}^{-t} \cos t & \mathrm{e}^{-t} \sin t \\ \mathrm{e}^{-t} \sin t & -\mathrm{e}^{-t} \cos t \end{pmatrix} \\ F &= \begin{pmatrix} \mathrm{e}^t \cos t \\ \mathrm{e}^t \sin t \end{pmatrix} \\ \Phi^{-1} F &= \begin{pmatrix} \mathrm{e}^{-t} \cos t & \mathrm{e}^{-t} \sin t \\ \mathrm{e}^{-t} \sin t & -\mathrm{e}^{-t} \cos t \end{pmatrix} \begin{pmatrix} \mathrm{e}^t \cos t \\ \mathrm{e}^t \sin t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \int \Phi^{-1} F dt &= \int \begin{pmatrix} 1 \\ 0 \end{pmatrix} dt = \begin{pmatrix} t \\ 0 \end{pmatrix} \\ X_p = \Phi \int \Phi^{-1} F dt &= \begin{pmatrix} \mathrm{e}^t \cos t & \mathrm{e}^t \sin t \\ \mathrm{e}^t \sin t & -\mathrm{e}^t \cos t \end{pmatrix} \begin{pmatrix} t \\ 0 \end{pmatrix} = \begin{pmatrix} t\,\mathrm{e}^t \cos t \\ t\,\mathrm{e}^t \sin t \end{pmatrix}. \end{aligned}$$

Putting this all together, we obtain a general solution to the system.

$$\boxed{X(t) = c_1 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \mathrm{e}^t + c_2 \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix} \mathrm{e}^t + \begin{pmatrix} t\,\mathrm{e}^t \cos t \\ t\,\mathrm{e}^t \sin t \end{pmatrix}}$$

◊

4 Laplace Transforms

Q20. $y' + 2y = \begin{cases} t, & 0 \le t < 1 \\ 0, & 1 \le t \end{cases}$; $y(0) = 0$.

Solution. We must first write the righthand side as a sum of unit-step function.

$$\begin{aligned}
f(t) &= \begin{cases} t, & 0 \le t < 1 \\ 0, & 1 \le t \end{cases} \\
f(t) &= t[\mathcal{U}(t) - \mathcal{U}(t-1)] \\
&= t\mathcal{U}(t) - t\mathcal{U}(t-1) \\
&= t\mathcal{U}(t) - (t-1+1)\mathcal{U}(t-1) \\
&= t\mathcal{U}(t) - (t-1)\mathcal{U}(t-1) - \mathcal{U}(t-1).
\end{aligned}$$

We now take the Laplace transform of both sides of the equation.

$$\begin{aligned}
[sY - y(0)] + 2Y &= \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s} \\
(s+2)Y &= \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s} \\
Y &= \frac{1}{s^2(s+2)} - \frac{e^{-s}}{s^2(s+2)} - \frac{e^{-s}}{s(s+2)}.
\end{aligned}$$

We now do partial fractions twice: once for $\frac{1}{s^2(s+2)}$ and once for $\frac{1}{s(s+2)}$.

$$\begin{aligned}
\frac{1}{s(s+2)} &= \frac{A}{s} + \frac{B}{s+2} \\
1 &= As + 2A + Bs \\
A &= \frac{1}{2} \\
B &= -\frac{1}{2}.
\end{aligned}$$

$$\begin{aligned}
\frac{1}{s^2(s+2)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} \\
1 &= As^2 + 2As + Bs + 2B + Cs^2 \\
A &= -\frac{1}{4} \\
B &= \frac{1}{2} \\
C &= \frac{1}{4}.
\end{aligned}$$

Putting these together and taking the inverse Laplace transform, we obtain the solution to the given differential equation.

$$Y = -\frac{1}{4}\left(\frac{1}{s}\right) + \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{1}{4}\left(\frac{1}{s+2}\right) - \frac{1}{4}\left(\frac{e^{-s}}{s}\right) - \frac{1}{2}\left(\frac{e^{-s}}{s^2}\right) + \frac{1}{4}\left(\frac{e^{-s}}{s+2}\right)$$

$$\boxed{y(t) = -\frac{1}{4} + \frac{1}{2}t + \frac{1}{4}e^{-2t} - \frac{1}{4}\mathcal{U}(t-1) - \frac{1}{2}(t-1)\mathcal{U}(t-1) + \frac{1}{4}e^{-2(t-1)}\mathcal{U}(t-1)}$$

◊

Q21. $y' = 1 - \sin t - \int_0^t y(\tau)d\tau$; $y(0) = 0$.

Solution. Take the Laplace transform of both sides.

$$\begin{aligned}
sY - y(0) &= \frac{1}{s} - \frac{1}{s^2+1} - \mathcal{L}\{y * 1\} \\
sY &= \frac{1}{s} - \frac{1}{s^2+1} - \frac{1}{s}Y \\
\frac{s^2+1}{s}Y &= \frac{1}{s} - \frac{1}{s^2+1} \\
Y &= \frac{1}{s^2+1} - \frac{s}{(s^2+1)^2} \\
Y &= \frac{1}{s^2+1} - \frac{1}{2}\left(\frac{2s}{(s^2+1)^2}\right) \\
&\boxed{y(t) = \sin t - \frac{1}{2}t\sin t}
\end{aligned}$$

◇

Q22. $y'' + y = \delta\left(t - \frac{\pi}{2}\right) + \delta\left(t - \frac{3\pi}{2}\right)$; $y(0) = y'(0) = 0$.

Solution. Take the Laplace transform of both sides.

$$\begin{aligned}
[s^2Y - sy(0) - y'(0)] + Y &= \mathrm{e}^{-(\pi/2)s} + \mathrm{e}^{-(3\pi/2)s} \\
(s^2+1)Y &= \mathrm{e}^{-(\pi/2)s} + \mathrm{e}^{-(3\pi/2)s} \\
Y &= \frac{\mathrm{e}^{-(\pi/2)s}}{s^2+1} + \frac{\mathrm{e}^{-(3\pi/2)s}}{s^2+1} \\
&\boxed{y(t) = \sin\left(t - \frac{\pi}{2}\right)\mathcal{U}\left(t - \frac{\pi}{2}\right) + \sin\left(t - \frac{3\pi}{2}\right)\mathcal{U}\left(t - \frac{3\pi}{2}\right)} \\
&\text{or} \\
&\boxed{y(t) = -\cos(t)\mathcal{U}\left(t - \frac{\pi}{2}\right) + \cos(t)\mathcal{U}\left(t - \frac{3\pi}{2}\right)}
\end{aligned}$$

◇

5 Power Series

Q23. $y'' - 2xy' + 8y = 0$; $y(0) = 3$, $y'(0) = 0$.

Solution. We search for a solution given by a power series centered at $x = 0$.

$$y = \sum_{n=0}^{\infty} c_n x^n;\ y' = \sum_{n=1}^{\infty} nc_n x^{n-1};\ y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 2x\sum_{n=1}^{\infty} nc_n x^{n-1} + 8\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 2\sum_{n=1}^{\infty} nc_n x^{n} + 8\sum_{n=0}^{\infty} c_n x^n = 0.$$

We now make appropriate substitutions for the indices on each sum so as to be left with x^k in the summand.

$$\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - 2\sum_{k=1}^{\infty} kc_k x^k + 8\sum_{k=0}^{\infty} c_k x^k = 0.$$

Next we combine the three sums into a single sum by rewriting each sum so they start at $k = 1$.

$$\left(2c_2 + \sum_{n=1}^{\infty}(k+2)(k+1)c_{k+2}x^k\right) - 2\sum_{k=1}^{\infty} kc_k x^k + 8\left(c_0 + \sum_{k=1}^{\infty} c_k x^k\right) = 0$$

$$2c_2 + 8c_0 + \sum_{n=1}^{\infty}[(k+2)(k+1)c_{k+2} - 2kc_k + 8c_k]x^k = 0.$$

We now make use of our initial conditions, which gives us $c_0 = y(0) = 3$ and $c_1 = y'(0) = 0$. When two polynomials are equal, their coefficients are equal. So we can use the constant term of our derived equation to determine the value of c_2.

$$\begin{aligned} 2c_2 + 8c_0 &= 0 \\ 2c_2 &= -8c_0 = -24 \\ c_2 &= -12. \end{aligned}$$

Finally, we use the power series portion of our derived equation to determine a recurrence relation which will allow us to solve for the remaining values of c_k.

$$\begin{aligned} (k+2)(k+1)c_{k+2} - 2kc_k + 8c_k &= 0 \\ c_{k+2} &= \frac{(2k-8)c_k}{(k+2)(k+1)}. \end{aligned}$$

This recurrence relation tells us that a term c_{k+2} depends on the value of c_k. Because $c_1 = 0$, we see that $c_3 = 0$, $c_5 = 0$, and hence for every odd index $2k+1$, $c_{2k+1} = 0$. We now focus on the even indices.

$$\begin{aligned} c_4 &= \frac{(2\cdot 2 - 8)c_2}{(2+2)(2+1)} = \frac{-4c_2}{4\cdot 3} = \frac{-c_2}{3} = 4 \\ c_6 &= \frac{(4\cdot 2 - 8)c_4}{(4+2)(4+1)} = \frac{0c_4}{6\cdot 4} = 0. \end{aligned}$$

By the same reasoning, we now see that if $k \geq 3$, $c_{2k} = 0$. Putting this all together, we get a solution to the given differential equation.

$$\boxed{y(x) = 3 - 12x^2 + 4x^4}$$

◇

Q24. $(x-1)y''-xy'+y=0$; $y(0)=-2$, $y'(0)=6$.

Solution. We search for a solution given by a power series centered at $x=0$.

$$y=\sum_{n=0}^{\infty}c_nx^n;\ y'=\sum_{n=1}^{\infty}nc_nx^{n-1};\ y''=\sum_{n=2}^{\infty}n(n-1)c_nx^{n-2}$$

$$(x-1)\sum_{n=2}^{\infty}n(n-1)c_nx^{n-2}-x\sum_{n=1}^{\infty}nc_nx^{n-1}+\sum_{n=0}^{\infty}c_nx^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)c_nx^{n-1}-\sum_{n=2}^{\infty}n(n-1)c_nx^{n-2}-\sum_{n=1}^{\infty}nc_nx^n+\sum_{n=0}^{\infty}c_nx^n=0.$$

We now make appropriate substitutions for the indices on each sum so as to be left with x^k in the summand.

$$\sum_{k=1}^{\infty}(k+1)kc_{k+1}x^k-\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k-\sum_{k=1}^{\infty}kc_kx^k+\sum_{k=0}^{\infty}c_kx^k=0.$$

Next we combine the four sums into a single sum by rewriting each sum so they start at $k=1$.

$$\sum_{k=1}^{\infty}(k+1)kc_{k+1}x^k-\left(2c_2+\sum_{k=1}^{\infty}(k+2)(k+1)c_{k+2}x^k\right)-\sum_{k=1}^{\infty}kc_kx^k+\left(c_0+\sum_{k=1}^{\infty}c_kx^k\right)=0$$

$$-2c_2+c_0+\sum_{k=1}^{\infty}[(k+1)kc_{k+1}-(k+2)(k+1)c_{k+2}-kc_k+c_k]x_k=0.$$

We now make use of our initial conditions, which gives us $c_0=y(0)=-2$ and $c_1=y'(0)=6$. When two polynomials are equal, their coefficients are equal. So we can use the constant term of our derived equation to determine the value of c_2.

$$\begin{aligned}-2c_2+c_0&=0\\ -2c_2&=-c_0=2\\ c_2&=-1.\end{aligned}$$

Finally, we use the power series portion of our derived equation to determine a recurrence relation which will allow us to solve for the remaining values of c_k.

$$\begin{aligned}(k+1)kc_{k+1}-(k+2)(k+1)c_{k+2}-kc_k+c_k&=0\\ c_{k+2}&=\frac{(k+1)kc_{k+1}-(k-1)c_k}{(k+2)(k+1)}.\end{aligned}$$

We use this recurrence relation to calculate the next few values of c_k to determine a pattern.

$$\begin{aligned}c_3&=\frac{2c_2-0c_1}{6}=\frac{c_2}{3}=-2\left(\frac{1}{3!}\right)\\ c_4&=\frac{6c_3-c_2}{12}=\frac{-12\left(\frac{1}{3!}\right)+2\left(\frac{1}{2!}\right)}{12}=\frac{-4\left(\frac{1}{3!}\right)+2\left(\frac{1}{3!}\right)}{4}=-2\left(\frac{1}{4!}\right)\\ c_n&=-2\left(\frac{1}{n!}\right).\end{aligned}$$

Putting this all together, we get a solution to the given differential equation.

$$\boxed{y(x)=-2\left(1+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right)+6x=8x-2\,\mathrm{e}^x}$$

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