Ex1; $y'' - 8y' + 12y = 40\sin(2x)$ $\begin{cases} y(0) = 5/2 \\ y'(0) = 4 \end{cases}$

A) Solution homogène: $\begin{Bmatrix} y'' = \ell^2 \\ y' = \ell \\ y = 1 \end{Bmatrix}$

$\underbrace{\ell^2}_{ⓐ} - \underbrace{8\ell}_{ⓑ} + \underbrace{12}_{ⓒ} = 0$

$\Delta = b^2 - 4AC \Longrightarrow \Delta = 8^2 - 4 \times 1 \times 12 = 16$ ou 4^2

$\ell_{1,2} = \dfrac{-b \pm \sqrt{\Delta}}{2a} \Longrightarrow \ell_{1,2} = \dfrac{-(-8) \pm \sqrt{16}}{2 \times 1} \begin{matrix} \nearrow 6 \\ \searrow 2 \end{matrix}$ $\boxed{\Delta > 0}$

$y_h = C_1^{te} e^{6x} + C_2^{te} e^{2x}$

B) Solution particulière:

$y_p = A\sin(2x) + B\cos(2x)$ } Dériver,

$y'_p = 2A\cos(2x) - 2B\sin(2x)$ } Dériver,

$y''_p = -4A\sin(2x) - 4B\cos(2x)$

$y_p = 12[A\sin(2x) + B\cos(2x)] = 12A\sin(2x) + 12B\cos(2x)$

$y'_p = -8[2A\cos(2x) - 2B\sin(2x)] = -16A\cos(2x) + 16B\sin(2x)$

$y''_p = 1[-4A\sin(2x) - 4B\cos(2x)] = -4A\sin(2x) - 4B\cos(2x)$

$\rightarrow 8A\sin(2x) - 16A\cos(2x) + 8B\cos(2x) + 16B\sin(2x) = 40\sin(2x)$

"Sin" $\rightarrow 8A + 16B = 40$ ①

"Cos" $\rightarrow -16A + 8B = 0$ ②

Système $\begin{cases} 8A + 16B = 40 & ① \\ -8A + 4B = 0 & ② \div 2 \end{cases}$ (+)

$0 + 20B = 40$

$B = 2$

$8A + 16(2) = 40$

$A = \dfrac{40 - 32}{8}$

$A = 1$

$y_p = [(1)\sin(2x)] + [(2)\cos(2x)]$

$y_p = \sin(2x) + 2\cos(2x)$

C) Solution générale:

$g(x) = y_h + y_p$

$g(x) = C_1^{te} e^{6x} + C_2^{te} e^{2x} + \sin(2x) + 2\cos(2x)$

D) Solution unique: (1) $y(0) = 5/2$ et (2) $y'(0) = 4$.

$$y(x) = C_1^{te} e^{6x} + C_2^{te} e^{2x} + Sin(2x) + 2Cos(2x)$$
$$y'(x) = 6C_1^{te\,6x} + 2C_2^{te} e^{2x} + 2Cos(2x) - 4Sin(2x)$$

(1) $$y(0) = C_1^{te} \underbrace{e^{6(0)}}_{1} + C_2^{te} \underbrace{e^{2(0)}}_{1} + \underbrace{Sin(2(0))}_{0} + 2\underbrace{Cos(2(0))}_{1} = \frac{5}{2}$$
$$= C_1^{te} + C_2^{te} + 0 + 2 = \frac{5}{2}$$
$$= C_1^{te} + C_2^{te} = \frac{1}{2}$$

(2) $$y'(0) = 6\underbrace{C_1^{te\,6(0)}}_{1} + 2\underbrace{C_2^{te\,2(0)}}_{1} + 2\underbrace{Cos(2(0))}_{1} - 4\underbrace{Sin(2(0))}_{0} = 4$$
$$= 6C_1^{te} + 2C_2^{te} + 2 = 4$$
$$= 6C_1^{te} + 2C_2^{te} = 2$$
$$= 3C_1^{te} + C_2^{te} = 1$$

Système $C_1^{te} = A$, $C_2^{te} = B$

(1) $A + B = 1/2$
(2) $3A + B = 1$

$(1/4) + B = 1/2$
$B = 1/2 - 1/4 = \frac{1}{4}$

(1) $A + B = 1/2$
(-2) $-3A - B = -1$

$-2A + 0 = -1/2$
$A = 1/4$

Test
$$3\left(\frac{1}{4}\right) + 1\left(\frac{1}{4}\right) = 1$$
$$\frac{3}{4} + \frac{1}{4} = 1$$
$$1 = 1$$

Donc $C_1^{te} = 1/4$ et $C_2^{te} = 1/4$

$$y_U = 1/4\, e^{6x} + 1/4\, e^{2x} + Sin(2x) + 2Cos(2x)$$

Ex 2 $y'' - 8y' + 12y = 40e^{2x}$ $\begin{cases} y(0) = 1 \\ y'(0) = -2 \end{cases}$

A) Solution homogène: $\begin{Bmatrix} y'' = \ell^2 \\ y' = \ell \\ y = 1 \end{Bmatrix}$

$\underbrace{\ell^2}_{a} - \underbrace{8\ell}_{b} + \underbrace{12}_{c} = 0$

$\Delta = b^2 - 4ac \Rightarrow \Delta = 8^2 - 4\times1\times12 = 16$

$\ell_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} \Rightarrow \ell_{1,2} = \frac{-(-8) \pm \sqrt{16}}{2\times1} \begin{cases} 6 \\ 2 \end{cases}$ $\boxed{\Delta > 0}$

$y_h = C_1^{te} e^{6x} + C_2^{te} e^{2x}$

B) Solution particulière

$\left.\begin{array}{l} y_p = Ae^{2x} \\ y'_p = 2Ae^{2x} \\ y''_p = 4Ae^{2x} \end{array}\right\}$ $\begin{array}{r} 1(y''_p) - 8(y'_p) + 12(y_p) = 40e^{2x} \\ 4Ae^{2x} - 16Ae^{2x} + 12Ae^{2x} = 40e^{2x} \\ 0 = 40e^{2x} \end{array}$

Pas possible d'avoir un système $\Rightarrow$ augmenter l'ordre:

$y_p = Axe^{2x}$

$y'_p = 2Axe^{2x} + Ae^{2x}$

$y''_p = 4Axe^{2x} + 2Ae^{2x} + 2Ae^{2x} = 4Axe^{2x} + 4Ae^{2x}$

$1(y''_p) - 8(y'_p) + 12(y_p) = 40e^{2x}$

$+\underbrace{4Axe^{2x}} + 4Ae^{2x} - \underbrace{16Axe^{2x}} - 8Ae^{2x} + \underbrace{12Axe^{2x}} = 40e^{2x}$

$0\cancel{Axe^{2x}} - 4A\cancel{e^{2x}} = 40\cancel{e^{2x}}$

$A = \frac{40}{-4} = -10$ $\boxed{A = -10}$

$y_p = Axe^{2x}$

$y_p = -10xe^{2x}$

C) Solution générale:

$y(x) = y_h + y_p$

$y(x) = C_1^{te} e^{6x} + C_2^{te} e^{2x} - 10xe^{2x}$

D/ Solution unique : ① $y(0)=1$ et ② $y'(0)=-2$

$$y(x) = C_1^{te} e^{6x} + C_2^{te} e^{2x} - 10x e^{2x}$$

$$y'(x) = 6C_1^{te} e^{6x} + 2C_2^{te} e^{2x} - 20x e^{2x} - 10 e^{2x}$$

(c)

$$① \; y(0) = C_1^{te} e^{6(0)} + C_2^{te} e^{2(0)} - 10(0) e^{2(0)} = 1$$

$$= C_1^{te}(1) + C_2^{te}(1) - (0) = 1$$

$$= C_1^{te} + C_2^{te} = 1$$

$$② \; y'(0) = 6C_1^{te} e^{6(0)} + 2C_2^{te} e^{2(0)} - 20(0) e^{2(0)} - 10 e^{2(0)} = -2$$

$$= 6C_1^{te}(1) + 2C_2^{te}(1) - (0) - 10(1) = -2$$

$$= 6C_1^{te} + 2C_2^{te} = -2 + 10 = 8$$

$$= 3C_1^{te} + C_2^{te} = 4$$

Système : $C_1^{te} = A$, $C_2^{te} = B$

$$\left.\begin{array}{l} ①\,;\; A + B = 1 \\ ②\,;\; 3A + B = 4 \end{array}\right] \quad \left.\begin{array}{l} ① \; A + B = 1 \\ (-2) \; -3A - B = -4 \end{array}\right\} \quad \begin{array}{l} \left(\frac{3}{2}\right) + B = 1 \\ B = 1 - \frac{3}{2} \\ B = \frac{-1}{2} \end{array}$$

$$2A - 0 = -3$$

$$A = \frac{3}{2}$$

Test

$$3\left(\frac{3}{2}\right) + \left(\frac{-1}{2}\right) = 4$$

$$\frac{9}{2} - \frac{1}{2} = 4$$

$$4 = 4$$

ok

$$C_1^{te} = \frac{3}{2}, \quad C_2^{te} = \frac{-1}{2}$$

$$y_u = \frac{3}{2} e^{6x} + \frac{-1}{2} e^{2x} - 10x e^{2x}$$

Ex3 $y'' - 8y' + 16y = 40e^{2x}$ $\begin{cases} y(0) = 2 \\ y'(0) = -2 \end{cases}$

A) Solution homogène: $\begin{bmatrix} y'' = \ell^2 \\ y' = \ell \\ y = 1 \end{bmatrix}$

$\underbrace{\ell^2}_{a} - 8\underbrace{\ell}_{b} + \underbrace{16}_{c} = 0$

$\Delta = b^2 - 4ac \Rightarrow \Delta = 8^2 - 4 \times 1 \times 16 = 0$ $\boxed{\Delta = 0}$

$\ell_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} \Rightarrow \ell_{1,2} = \frac{-(-8) \pm \sqrt{0}}{2 \times (1)} = 4.$

$y_h = C_1^{te} e^{4x} + C_2^{te} x e^{4x}$

B) Solution particulière:

$y_p = Ae^{2x}$
$y'_p = 2Ae^{2x}$
$y''_p = 4Ae^{2x}$

$1(y''_p) - 8(y'_p) + 16(y_p) = 40e^{2x}$

$4Ae^{2x} \cancel{-16Ae^{2x}} \cancel{+16Ae^{2x}} = 40e^{2x}$

$4A\cancel{e^{2x}} = 40\cancel{e^{2x}}$

$A = \frac{40}{4} = 10$

$y_p = Ae^{2x}$

$y_p = 10e^{2x}$

C) Solution générale:

$y(x) = y_h + y_p$

$y(x) = C_1^{te} e^{4x} + C_2^{te} x e^{4x} + 10e^{2x}$

D) Solution unique: ① $y(0) = 2$ et ② $y'(0) = -2$

$y'(x) = 4C_1^{te} e^{4x} + 4C_2^{te} x e^{4x} + \underbrace{C_2^{te} e^{4x}}_{c} + 20e^{2x}$

① $y(0) = C_1^{te} e^{4(0)} + C_2^{te}(0) e^{4(0)} + 10e^{2(0)} = 2$

$C_1^{te}(1) + (0) + 10(1) = 2 \Rightarrow C_1^{te} = -8$

② $y'(0) = 4(-8)e^{4(0)} + 4C_2^{te}(0)e^{4(0)} + C_2^{te} e^{4(0)} + 20e^{2(0)} = -2$

$= -32(1) + (0) + C_2^{te}(1) + 20 = -2 \Rightarrow C_2^{te} = 10$

$y_u = -8e^{4x} + 10xe^{4x} + 10e^{2x}$

Ex4: $y''-8y'+25y=40e^{2x}$ $\begin{cases} y(0)=0 \\ y'(0)=-4 \end{cases}$

A) Solution homogène: $\begin{Bmatrix} y''=\ell^2 \\ y'=\ell \\ y=1 \end{Bmatrix}$

$\ell^2-8\ell+25=0$

$\Delta=b^2-4ac \Rightarrow \Delta=8^2-4\times1\times25=-36$ $\boxed{\Delta<0}$

$\ell_{1,2}=\dfrac{-b\pm\sqrt{\Delta}}{2a} \Rightarrow \dfrac{-(-8)\pm\sqrt{36j^2}}{2\times1}=4\pm6j$

$y_h=e^{4x}\left[A\cos(6x)+B\sin(6x)\right]$

B) Solution particulière:

Dériver: $y_p=Ce^{2x}$

Dériver. $y'_p=2Ce^{2x}$

$y''_p=4Ce^{2x}$

$1(y''_p)-8(y'_p)+25(y_p)=40e^{2x}$

$4Ce^{2x}-16Ce^{2x}+25Ce^{2x}=40e^{2x}$

$13C\cancel{e^{2x}}=40\cancel{e^{2x}}$

$C=\dfrac{40}{13}$

$y_p=Ce^{2x}$

$y_p=\dfrac{40}{13}e^{2x}$

C) Solution générale:

$y(x)=y_h+y_b$

$y(x)=e^{4x}\left[A\cos(6x)+B\sin(6x)\right]+\dfrac{40}{13}e^{2x}$

D) Solution unique: ① $y(0)=0$ et ② $y'(0)=-4$

(dérivée de $y(x)$) $y'(x)=4e^{4x}\left[A\cos(6x)+B\sin(6x)\right]+e^{4x}\left[-6A\sin(6x)+6B\cos(6x)\right]+\dfrac{80}{13}e^{4x}$

① $y(0)=e^{4(0)}\left[A\cos(6(0))+B\sin(6(0))\right]+\dfrac{40}{13}e^{2(0)}=0$

$=(1)\left[A(1)+(0)\right]+\dfrac{40}{13}(1)=A+\dfrac{40}{13}=0 \Rightarrow A=\dfrac{-40}{13}$

② $y'(0)=4e^{4(0)}\left[\frac{-40}{13}\cos(6(0))+B\sin(6(0))\right]+e^{4(0)}\left[-6A\sin(6(0))+6B\cos(6(0))\right]+\dfrac{80}{13}e^{4(0)}=-4$

$=4(1)\left[\frac{-40}{13}(1)+(0)\right]+(1)\left[(0)+6B(1)\right]+\dfrac{80}{13}(1)=\dfrac{-160}{13}+6B+\dfrac{80}{13}=-4$

$B=\dfrac{14}{39}$

$y_U=e^{4x}\left[\frac{-40}{13}\cos(6x)+\frac{14}{39}\sin(6x)\right]+\dfrac{40}{13}e^{2x}$