FORMULAIRE RELATIF AUX OPÉRATEURS

Soient U et V deux champs scalaires et $\vec{a}$ et $\vec{b}$ deux champs vectoriels.

1. Formules portant sur un seul champ:

1. $\vec{\nabla}.(\vec{\nabla}U) = \vec{\nabla}^2 U$ soit $\mathrm{div}(\overrightarrow{\mathrm{grad}}\, U) = \Delta U$

2. $\vec{\nabla} \wedge (\vec{\nabla}U) = \vec{0}$ soit $\overrightarrow{\mathrm{rot}}(\overrightarrow{\mathrm{grad}}\, U) = \vec{0}$

3. $\vec{\nabla}.(\vec{\nabla} \wedge \vec{a}) = \vec{0}$ soit $\mathrm{div}(\overrightarrow{\mathrm{rot}}\, \vec{a}) = \vec{0}$

4. $\vec{\nabla} \wedge (\vec{\nabla} \wedge \vec{a}) = \vec{\nabla}(\vec{\nabla}.\vec{a}) - \vec{\nabla}^2 \vec{a}$ soit $\overrightarrow{\mathrm{rot}}(\overrightarrow{\mathrm{rot}}\, \vec{a}) = \overrightarrow{\mathrm{grad}}(\mathrm{div}\, \vec{a}) - \Delta \vec{a}$

2. Formules portant sur deux champs:

5. $\vec{\nabla}(UV) = V\vec{\nabla}(U) + U\vec{\nabla}(V)$ soit $\overrightarrow{\mathrm{grad}}(UV) = V\,\overrightarrow{\mathrm{grad}}\, U + U\,\overrightarrow{\mathrm{grad}}\, V$

6. $\vec{\nabla}.(U\vec{a}) = \vec{a}(\vec{\nabla}U) + U(\vec{\nabla}.\vec{a})$ soit $\mathrm{div}(U\vec{a}) = \overrightarrow{\mathrm{grad}}\, U.\vec{a} + U\,\mathrm{div}\,\vec{a}$

7. $\vec{\nabla} \wedge (U\vec{a}) = (\vec{\nabla}U) \wedge \vec{a} + U(\vec{\nabla} \wedge \vec{a})$ soit $\overrightarrow{\mathrm{rot}}(U\vec{a}) = \overrightarrow{\mathrm{grad}}\, U \wedge \vec{a} + U\,\overrightarrow{\mathrm{rot}}\,\vec{a}$

8. $\vec{\nabla}.(\vec{a} \wedge \vec{b}) = \vec{b}.(\vec{\nabla} \wedge \vec{a}) - \vec{a}.(\vec{\nabla} \wedge \vec{b})$ soit $\mathrm{div}(\vec{a} \wedge \vec{b}) = \vec{b}.\overrightarrow{\mathrm{rot}}\,\vec{a} - \vec{a}.\overrightarrow{\mathrm{rot}}\,\vec{b}$

9. $\vec{\nabla} \wedge (\vec{a} \wedge \vec{b}) = (\vec{\nabla}.\vec{b})\vec{a} - (\vec{\nabla}.\vec{a})\vec{b} + (\vec{b}.\vec{\nabla})\vec{a} - (\vec{a}.\vec{\nabla})\vec{b}$

soit $\overrightarrow{\mathrm{rot}}(\vec{a} \wedge \vec{b}) = (\mathrm{div}\,\vec{b})\vec{a} - (\mathrm{div}\,\vec{a})\vec{b} + (\vec{b}\,\overrightarrow{\mathrm{grad}})\vec{a} - (\vec{a}.\overrightarrow{\mathrm{grad}})\vec{b}$

10. $\vec{\nabla}(\vec{a}.\vec{b}) = \vec{a} \wedge (\vec{\nabla} \wedge \vec{b}) + \vec{b} \wedge (\vec{\nabla} \wedge \vec{a}) + (\vec{b}.\vec{\nabla})\vec{a} + (\vec{a}.\vec{\nabla})\vec{b}$

soit $\overrightarrow{\mathrm{grad}}(\vec{a}.\vec{b}) = \vec{a} \wedge (\overrightarrow{\mathrm{rot}}\,\vec{b}) + \vec{b} \wedge (\overrightarrow{\mathrm{rot}}\,\vec{a}) + (\vec{b}.\overrightarrow{\mathrm{grad}})\vec{a} + (\vec{a}.\overrightarrow{\mathrm{grad}})\vec{b}$

3. Expressions des opérateurs dans divers systèmes de coordonnées:

a. Gradient:

* cartésiennes: $\overrightarrow{\mathrm{grad}}\, U = \left(\frac{\partial U}{\partial x}\right)\vec{e}_x + \left(\frac{\partial U}{\partial y}\right)\vec{e}_y + \left(\frac{\partial U}{\partial z}\right)\vec{e}_z$

* cylindriques: $\overrightarrow{\mathrm{grad}}\, U = \left(\frac{\partial U}{\partial r}\right)\vec{e}_r + \frac{1}{r}\left(\frac{\partial U}{\partial \theta}\right)\vec{e}_\theta + \left(\frac{\partial U}{\partial z}\right)\vec{e}_z$

* sphériques: $\overrightarrow{\mathrm{grad}}\, U = \left(\frac{\partial U}{\partial r}\right)\vec{e}_r + \frac{1}{r}\left(\frac{\partial U}{\partial \theta}\right)\vec{e}_\theta + \frac{1}{r\sin\theta}\left(\frac{\partial U}{\partial \varphi}\right)\vec{e}_\varphi$

b. Divergence:

* cartésiennes: $\mathrm{div}\,\vec{a} = \left(\frac{\partial a_x}{\partial x}\right) + \left(\frac{\partial a_y}{\partial y}\right) + \left(\frac{\partial a_z}{\partial z}\right)$

* cylindriques: $\mathrm{div}\,\vec{a} = \frac{1}{r}\left(\frac{\partial r a_r}{\partial r}\right) + \frac{1}{r}\left(\frac{\partial a_\theta}{\partial \theta}\right) + \left(\frac{\partial a_z}{\partial z}\right)$

* sphériques: $\mathrm{div}\,\vec{a} = \frac{1}{r^2}\left(\frac{\partial r^2 a_r}{\partial r}\right) + \frac{1}{r\sin\theta}\left(\frac{\partial a_\theta \sin\theta}{\partial \theta}\right) + \frac{1}{r\sin\theta}\left(\frac{\partial a_\varphi}{\partial \varphi}\right)$

c. Rotationnel:

$$\text{* cartésiennes: } \overrightarrow{\text{rot}}\,\vec{a} = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\vec{e}_x + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\vec{e}_y + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\vec{e}_z$$

$$\text{* cylindriques: } \overrightarrow{\text{rot}}\,\vec{a} = \left(\frac{1}{r}\frac{\partial a_z}{\partial \theta} - \frac{\partial a_\theta}{\partial z}\right)\vec{e}_r + \left(\frac{\partial a_r}{\partial z} - \frac{\partial a_z}{\partial r}\right)\vec{e}_\theta + \frac{1}{r}\left(\frac{\partial (r a_\theta)}{\partial r} - \frac{\partial a_r}{\partial \theta}\right)\vec{e}_z$$

* sphériques:

$$\overrightarrow{\text{rot}}\,\vec{a} = \frac{1}{r\sin\theta}\left(\frac{\partial (a_\varphi \sin\theta)}{\partial \theta} - \frac{\partial a_\theta}{\partial \varphi}\right)\vec{e}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial a_r}{\partial \varphi} - \frac{\partial (r a_\varphi)}{\partial r}\right)\vec{e}_\theta + \frac{1}{r}\left(\frac{\partial (r a_\theta)}{\partial r} - \frac{\partial a_r}{\partial \theta}\right)\vec{e}_\varphi$$

d. Laplacien:

$$\text{* cartésiennes: } \Delta U = \left(\frac{\partial^2 U}{\partial x^2}\right) + \left(\frac{\partial^2 U}{\partial y^2}\right) + \left(\frac{\partial^2 U}{\partial z^2}\right)$$

$$\text{* cylindriques: } \Delta U = \left(\frac{\partial^2 U}{\partial r^2}\right) + \frac{1}{r}\left(\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 U}{\partial \theta^2}\right) + \left(\frac{\partial^2 U}{\partial z^2}\right)$$

$$\text{* sphériques: } \Delta U = \left(\frac{\partial^2 U}{\partial r^2}\right) + \frac{2}{r}\left(\frac{\partial U}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2 U}{\partial \varphi^2}\right) + \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial U}{\partial \theta}\right)\right)$$

4. Action des opérateurs sur le trièdre de base:

a. coordonnées cylindriques:

$\overrightarrow{\text{rot}}\,\vec{e}_r = \vec{0}$; $\overrightarrow{\text{rot}}\,\vec{e}_\theta = \dfrac{\vec{e}_z}{r}$; $\overrightarrow{\text{rot}}\,\vec{e}_z = \vec{0}$ pour le rotationnel

$\text{div}\,\vec{e}_r = \dfrac{1}{r}$; $\text{div}\,\vec{e}_\theta = 0$; $\text{div}\,\vec{e}_z = 0$ pour la divergence

b. coordonnées sphériques:

$\overrightarrow{\text{rot}}\,\vec{e}_r = \vec{0}$; $\overrightarrow{\text{rot}}\,\vec{e}_\theta = \dfrac{\vec{e}_\varphi}{r}$; $\overrightarrow{\text{rot}}\,\vec{e}_\varphi = \dfrac{\cos\theta}{r\sin\theta}\vec{e}_r - \dfrac{\vec{e}_\theta}{r}$ pour le rotationnel

$\text{div}\,\vec{e}_r = \dfrac{2}{r}$; $\text{div}\,\vec{e}_\theta = \dfrac{1}{r\tan\theta}$; $\text{div}\,\vec{e}_\varphi = 0$ pour la divergence